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THE IMAGINARY POINTS OF GEOMETRY.

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1. Infinity and Imaginary Points. Geometry was at one time regarded as describing actual spacial relations in an absolute and incontrovertible manner. Many philosophers supposed that by a process of pure deduction one started from self-obvious logical facts and discovered the inner constitution of space. Popular interest in the discovery of the now classical non-Euclidean geometry has been due largely to the still bewildering novelty of the fact that at certain stages in the development of an axiomatic system, there is a feature of arbitrary choice.

The geometry of real finite points being assumed in accordance with Euclidean standards, certain alternatives are yet open. One of these alternatives is in regard to the conception of points at infinity. As is well known, we may consistently adopt any one of the following conventions and indeed other possibilities remain.

(*a*) All points are finitely accessible. In other words there are no points at infinity.

(*b*) The points at infinity for a space of n dimensions constitute a (projective) space of $n - 1$ dimensions. Thus the Euclidean plane is completed by a line at infinity, the Euclidean three-space by a plane at infinity, and so forth. This is the convention of projective geometry.

(*c*) There is a single point at infinity. This is the convention of the geometry of inversion.

(*d*) The points at infinity fall upon n linear $(n - 1)$-spaces. Thus the Euclidean plane is completed by two intersecting straight lines at infinity, the Euclidean space by three planes at infinity and so forth. This is the convention of what has been called by some writers "function space."

The character of the set of points at infinity having been disposed of, the question of imaginary points is still open. The two common alternatives are as follows.

(*a*) To exclude from the discussion the consideration of imaginary points, or in other words to assume the existence of real points only.

(*b*) To assume the existence of points whose Cartesian coördinates are complex numbers as used in algebra.

So far as the writer is aware no other alternative has been currently suggested.

For problems of analysis situs and for investigations confined to a single net of rationality[1], it is assuredly permissible to assume the non-existence of imaginary points. In enumerative geometry, on the other hand, one has constant use for the so-called "fundamental law of algebra." The theorem that two curves, one of the mth order, and the other of the nth order, intersect in mn points not necessarily distinct, and like theorems, form the basis of this branch of mathematical science. For enumerative geometry, therefore, the convention that imaginary points exist with complex algebraic numbers for coördinates is essential.

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[1] Cf., for example, Veblen and Young, *Projective Geometry*, vol. 1, page 84.

But there are still other types of geometrical investigation and such as lead quite as naturally to a different method of extending the set of real points, and the present paper is intended to suggest some thoughts along this line.

Most mathematicians are familiar with the extensive use of complex numbers of the form $a + ib$, where $i^2 = -1$, in the study of electrical alternating current phenomena. A single algebraic variable is made to handle an essentially two-dimensional problem. In a similar manner, the methods of Hamilton are devised to treat a four-dimensional problem by the use of a single symbol, the quaternion, of the form $b = c_0 + ic_1 + jc_2 + kc_3$. The range of variation of two independent quarternions may be spoken of as a *two-space* or *plane*. A point in the plane will be given when particular values are assigned to the coördinates $(x_0 + ix_1 + jx_2 + kx_3,\ y_0 + iy_1 + jy_2 + ky_3)$. Here x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, are eight real quantities in the ordinary sense.

The analysis in such a case differs somewhat from that for the usual complex numbers, since for quaternions multiplication is not in general commutative. Despite this fact one is occasionally obliged to investigate a problem in two quaternions, and the free use of geometric language serves here as elsewhere to render the results easily intelligible. This corresponds merely to speaking of a quaternion as a generalized number. A point in the plane will be called real, however, only if x_1, x_2, x_3, y_1, y_2, y_3 are simultaneously zero. The set of real points constitutes a real plane in the usual sense. The quaternion $b = c_0 + ic_1 + jc_2 + kc_3$ is from this point of view, real, when $c_1 = c_2 = c_3 = 0$, and "complex," or better, "hypercomplex" in other cases. The quaternions, $b = c_0 + ic_1 + jc_2 + kc_3$ and $\bar{b} = c_0 - ic_1 - jc_2 - kc_3$ are "conjugates," so that $b\bar{b} = \bar{b}b = c_0^2 + c_1^2 + c_2^2 + c_3^2$.

2. Vector Analysis. When we approach geometry from the standpoint of vector analysis, we find certain operations and rules of combination to be of great significance and others to be apparently of but slight importance. We shall illustrate our future remarks by reference to a space of three dimensions although all of our conclusions will be readily extended to linear spaces of any desired dimensionality.

Vectors are interpreted as points by the artifice of introducing a fixed origin, and of regarding any point as the terminal point of a vector originating from this fixed origin. Vectors and hence points are capable of unrestricted combination by addition and subtraction. Two vectors and hence two points, say a and a', determine a scalar called their inner product $a \cdot a'$. Any point may be multiplied by any scalar. Scalars in the ordinary theory are merely numbers and are called scalars to emphasize their distinction from vectors. Two or more points have furthermore an outer product that may be expressed by a determinant, when the given points are referred to any system of Cartesian coördinates having the given origin. The square of the distance between two points a and a' is a scalar expressible as $(a - a') \cdot \overline{(a - a')}$, where the second factor is the "conjugate" of the first. Hence the point a' lies on the surface of a sphere of center a and radius r, if $(a - a')\cdot\overline{(a - a')} = r^2$. A fundamental property of vectors and hence of points in this system, is the following. If a and a' be any two points, $\surd\,(a\cdot\bar{a}) + \surd\,(a'\cdot\bar{a}') \geqq \surd\,[(a - a')\cdot\overline{(a - a')}]$. In other words the sum of the lengths of two sides of a triangle is always greater than the length of the third side. It is to be noted that this inequality serves to give a meaning to certain infinite processes by making available a definition of limits among vectors.[1]

In extending a real system so as to include also imaginary points, a student versed in vector analysis is inclined to insist upon the preservation of such vectorial properties only as have proved most significant. To be sure, simple examples will always arise in which the imaginary points may be dispensed with. But when problems occur analogous to those with which vector analysis usually deals it is always a matter of interest to determine whether the customary methods are applicable or whether special technique must be developed. The outer product of two vectors is not commutative, and to sacrifice the commutative character of the inner product is not out of the question.

In the generalization to be considered, a point in three-space will continue to be designated by an ordered set of three non-homogeneous scalar coördinates (a_1, a_2, a_3). By this means the notation current for real points is preserved in the generalization, and the difficulties are referred a step further back, namely to the notion of scalar. Each coördinate is called a scalar or *generalized number* and is such as to reduce in a special case to a real number. We shall represent each of these scalars, a, in terms of more elementary quantities, b, in the following manner: Each a is a square matrix of order m, having the elements of the principal diagonal equal to each other, and elements below this principal diagonal, throughout equal to zero, while elements above this principal diagonal are left unrestricted. Any two scalars, a_1 and a_2, will be combined for purposes of addition and multiplication in accordance with the usual rules for adding and multiplying matrices, the multiplication for a pair of these square matrices being by rows of the first and columns of the second factor. The product of two generalized numbers is again a generalized number, but this product is not commutative unless further restrictions be imposed. Explicitly, let

$$a \equiv \left|\begin{array}{ccccc} b_0, & b_{12}, & b_{13}, & \cdots, & b_{1m} \\ 0\,, & b_0\,, & b_{23}, & \cdots, & b_{2m} \\ 0\,, & 0\,, & b_0\,, & \cdots, & b_{3m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0\,, & 0\,, & 0\,, & \cdots, & b_0 \end{array}\right| .$$

The scalar, a, will be said to be "real" when and only when all of the elements above the principal diagonal vanish and when furthermore the single remaining independent element, namely the common element in the principal diagonal, is itself "real," as this term will be later defined. By the *conjugate* of a, i.e., $\bar{a}$, is meant the scalar $b_0\bar{b}_0/a$, and it is defined only for such scalars as have $b_0 \neq 0$.

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[1] The importance of an inequality such as here illustrated for the establishment of limits, is well developed by J. Kürschák, "Über Limesbildung und allgemeine Körpertheorie", *Journal für die reine und angewandte Mathematik*, vol. 142, 1913, pp. 211-253.

We may use the term *absolute value,* in connection with a scalar, a, and denote it by the symbol, $|a|$. This we define in terms of simpler elements, putting $|a|$ equal to $|b_0|$, where b_0 is the common diagonal element of the matrix defining the scalar, a, as above. As may be verified presently, the absolute value of any element, b, is so defined that the absolute value of a sum is equal to or less than the sum of the absolute values, and the absolute value of a product is equal to the product of the absolute values. The same propositions, therefore, will hold for the absolute value as applied to scalars, a, in fact $|a|^2 = a\bar{a}$, whenever $\bar{a}$ is defined. The distance from the origin to the point (a_1, a_2, a_3) is defined as the square root of the sum of the squares of the absolute values of the several coordinates. This is also the length of the vector from the origin to the point discussed. The question of signs arises here, in connection with the square root, in precisely the same manner that it does for real points. More generally, the inner product of the two points (a_1, a_2, a_3) and (a_1', a_2', a_3') in this order is defined as the scalar $a_1a_1' + a_2a_2' + a_3a_3'$. The sum of two vectors being defined in the ordinary manner, we shall have as a general theorem that the length of the vector obtained as the sum of two given vectors is equal to or less than the sum of the lengths of the two component vectors.

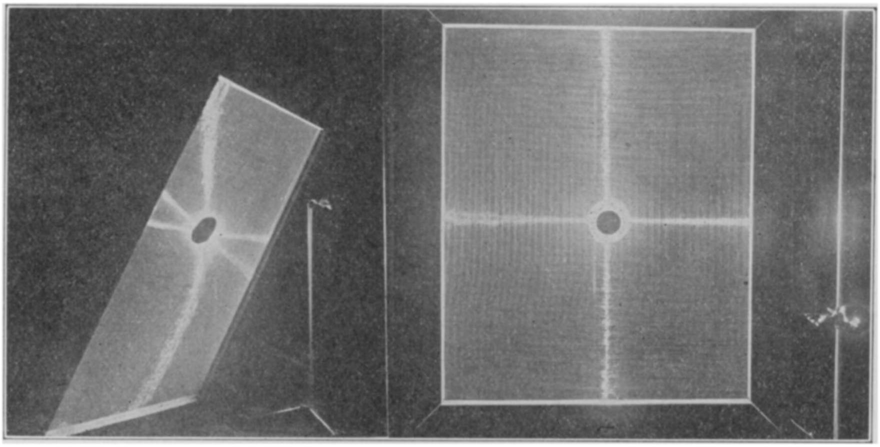
3. **Quaternions as Elements.** It remains to identify the elements, b, in terms of known quantities. One possibility is to consider the b's as themselves ordinary real numbers. A more interesting possibility has been suggested already, namely, each b may be taken as the quaternion expressed in the usual form, $b = c_0 + ic_1 + jc_2 + kc_3$, where c_0, c_1, c_2, c_3, are all ordinary real numbers, called the "fundamental components" of the quaternion b. If in particular, a subset be considered where the coefficients of j and k are zero for each of the quaternions, b, in question, these quaternions become merely ordinary complex numbers of algebra, or at least may be so regarded. If further, the order, m, of the matrices a, be unity, the scalars, a, will be themselves ordinary complex numbers.

Equations in vector analysis with real coefficients may frequently be interpreted in terms of these generalized numbers and so refer as well to imaginary points of geometry, different from those usually treated. When m, the order of the square matrix used in defining the "scalar," a, is greater than unity, we are confronted with the difficulty that the conjugate of a sum is not in general the sum of the conjugates of the given quantities. Thus the rationalization of even so simple an equation as that of the spheroid whose bipolar equation may be written $\sqrt{[(x - a)\cdot\overline{(x - a)}]} + \sqrt{[(x - a')\cdot\overline{(x - a')}]} = \sqrt{(d\cdot\bar{d})}$ presents difficulties in the general case. Other inconvenient features are likely to impress any one more interested in the simplicity of handling equations than in the generality of the results secured. However, considerations of this sort are of interest, since, on the one hand, they show the possibilities of other interpretations of equations which have long been studied and, on the other hand, indicate the availability of geometrical language in domains which might, at first, seem unsuited to such phraseology. If the order, m, of the matrices representing generalized numbers be allowed to become infinite in an appropriate manner, these generalized numbers become operators of the form used in the theory of integral equations. While it has not been customary to study integral equations in the domain of quaternions, some interesting results might be obtained in such a case.

LINES OF ILLUMINATION CAUSED BY THE PASSAGE OF LIGHT THROUGH A SCREEN.[1]

By WM. H. ROEVER, Washington University.

1. **Description.** On viewing a point source of light through a screen one will see on the screen several streamers of light emanating from the point in which the plane of the screen is pierced by the line connecting the eye with the light. The number of streamers which appear depends upon the direction of this line with respect to the plane of the screen. If, in particular, this line is perpendicular to the plane of the screen, there are four streamers, or two lines of light which cut at right angles. This is also the case for some other positions of the light and eye. In general, however, there are six streamers of light. On an old screen, these streamers are short and straight and one gets the impression of seeing a four- or six-pointed star. This is especially noticeable on looking at a source of light through the dust screen of a sleeping-car window. At first, when the source of light is far ahead of the train, the star will appear to have six points, but when the train is opposite the source of light the star appears to have only four points. If the screen is new and is made of polished wires of brass or copper, the streamers extend to the frame and are well-defined curves. Views of this phenomenon are shown in the accompanying photographic reproductions. The black spot in these



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[1] Presented to the American Mathematical Society, November 27, 1915. See *Bulletin* of that Society, vol. 22, p. 218.